Primordial black hole binaries as a probe of Hubble parameter

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Based on 2206.03142 & 2312.13728

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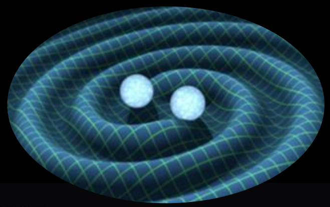
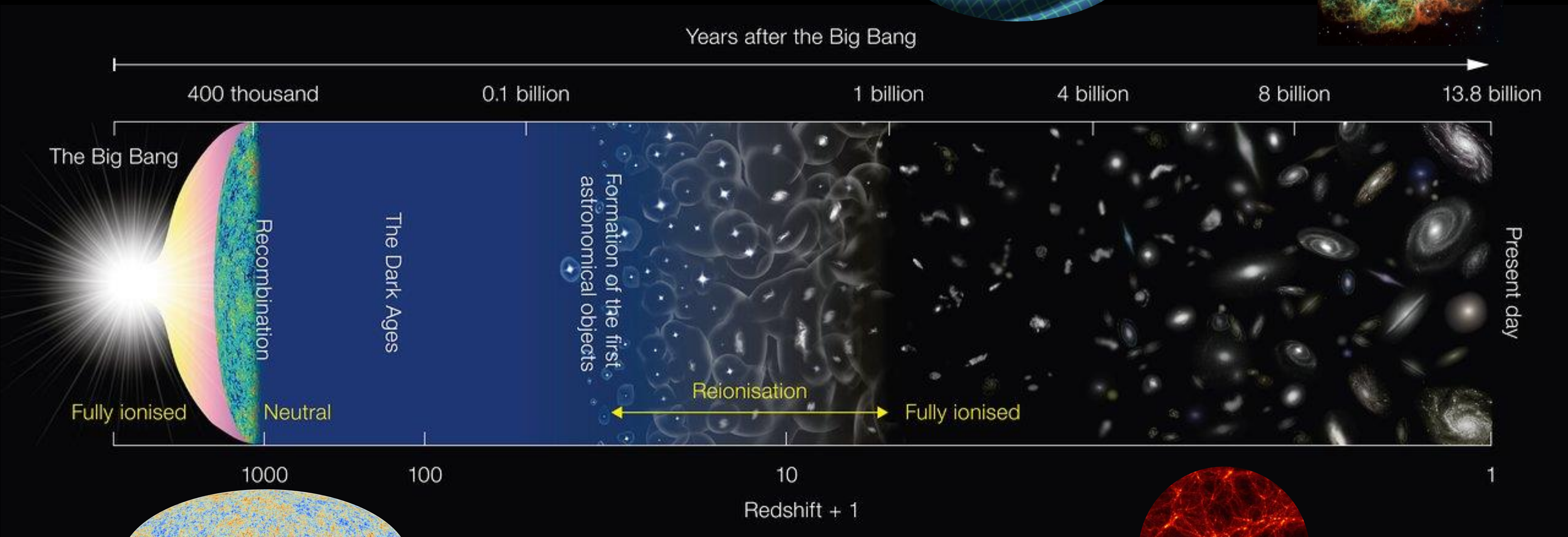
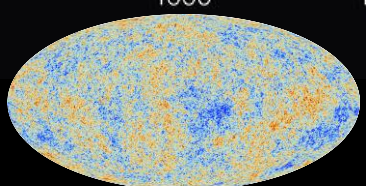
Center for Theoretical Physics of the Universe

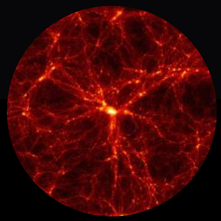
Cosmology, Gravity and Astroparticle Physics

73.04 ± 1.04 km/s/Mpc





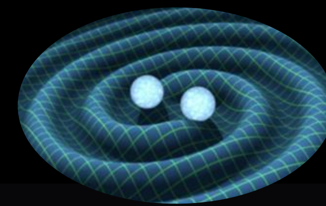


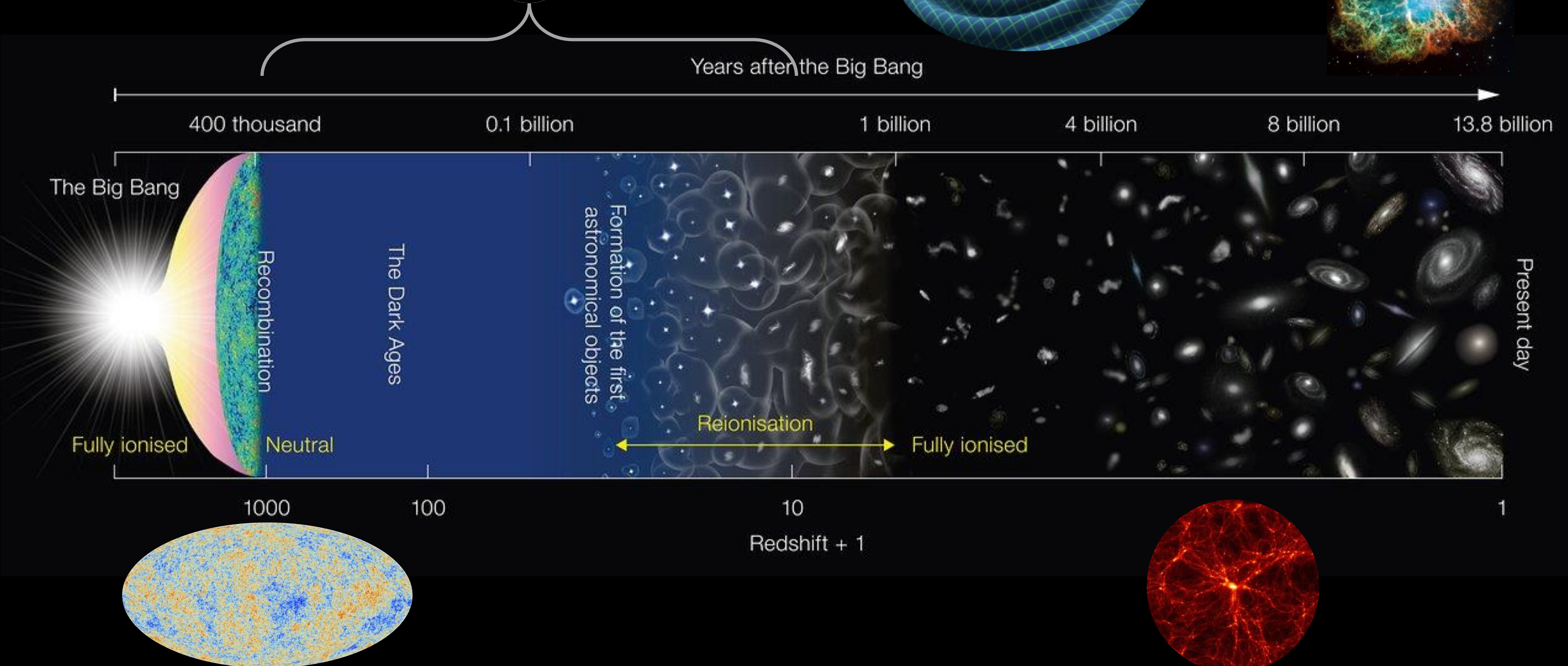


67.4 ± 0.5 km/s/Mpc

Image Credit: NAOJ

73.04 ± 1.04
km/s/Mpc





67.4 ± 0.5
km/s/Mpc

Image Credit: NAOJ

Primordial black holes as a potential candidate

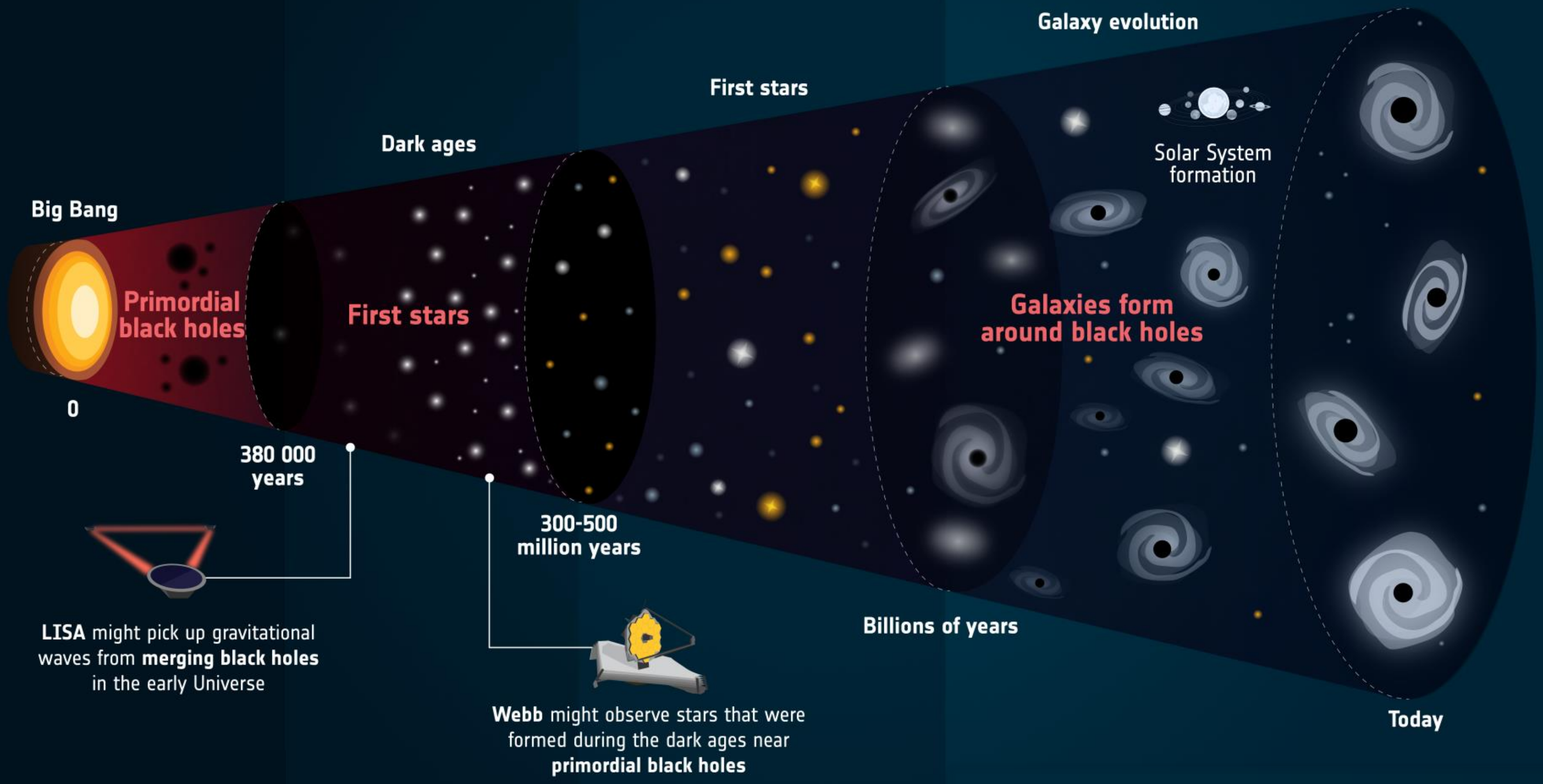


Image Credit: ESA

How to observe the signals from PBHs?

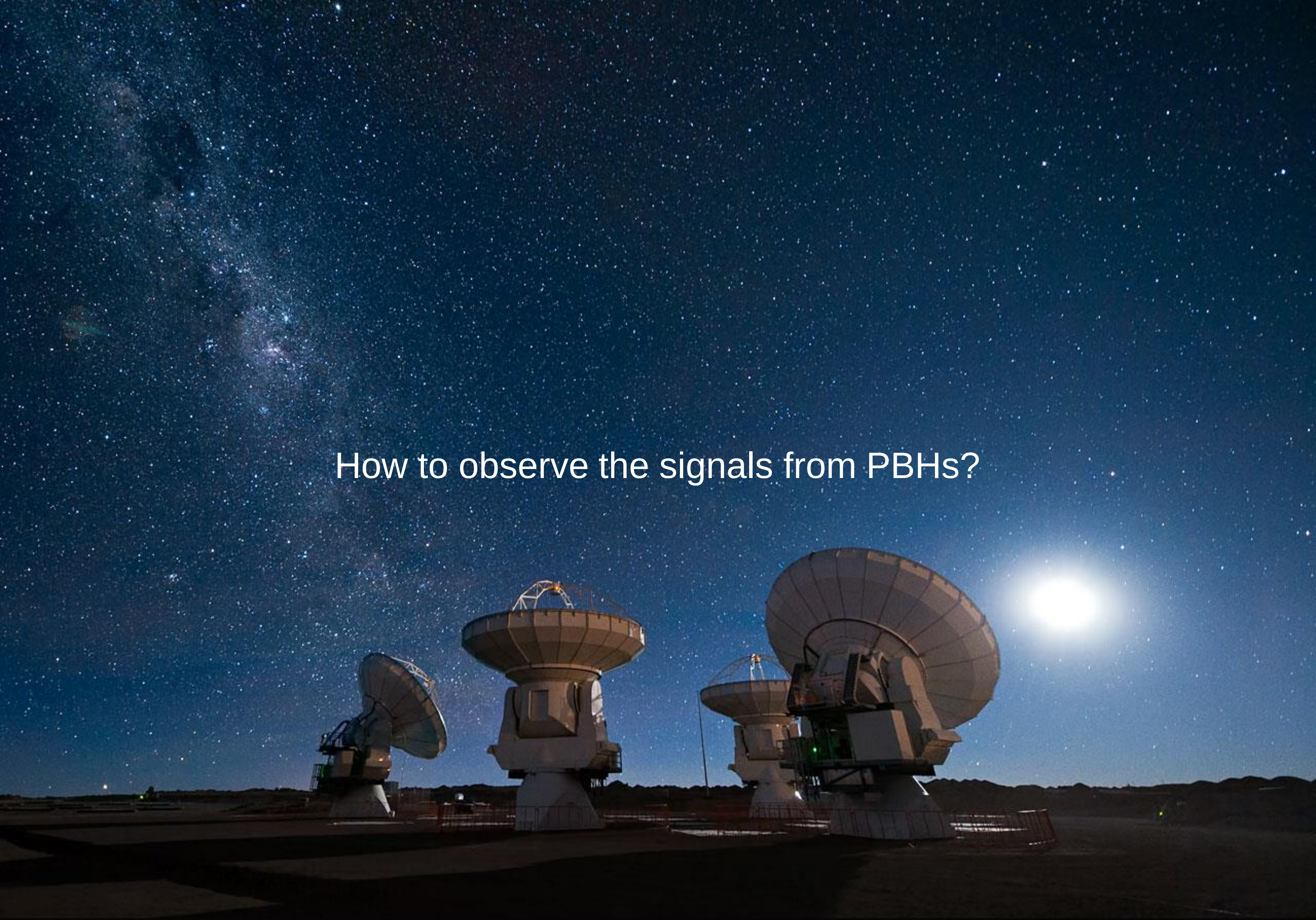


Image Credit: ESO

Image Credit: ESO

$$h(t) = \frac{4}{d_L(z)} \left(\frac{G\mathcal{M}_z}{c^2} \right)^{5/3} \left(\frac{\pi f(t)}{c} \right)^{2/3} \cos \Phi(t)$$

$$d_L = \frac{1+z}{H_0} \int_0^z \frac{c}{E(z')} dz' \qquad \mathcal{M}_z = \mathcal{M}(1+z)$$

Image Credit: ESO

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$$d_L = \frac{1+z}{H_0} \int_0^z \frac{c}{E(z')} dz' \qquad \mathcal{M}_z = \mathcal{M}(1+z)$$

$$\Delta t = \frac{1}{H_0} \int_{z_1}^{z_2} \frac{dz}{E(z)(1+z)}$$

Image Credit: ESO

How to construct redshift-distance (redshift-time) relation?

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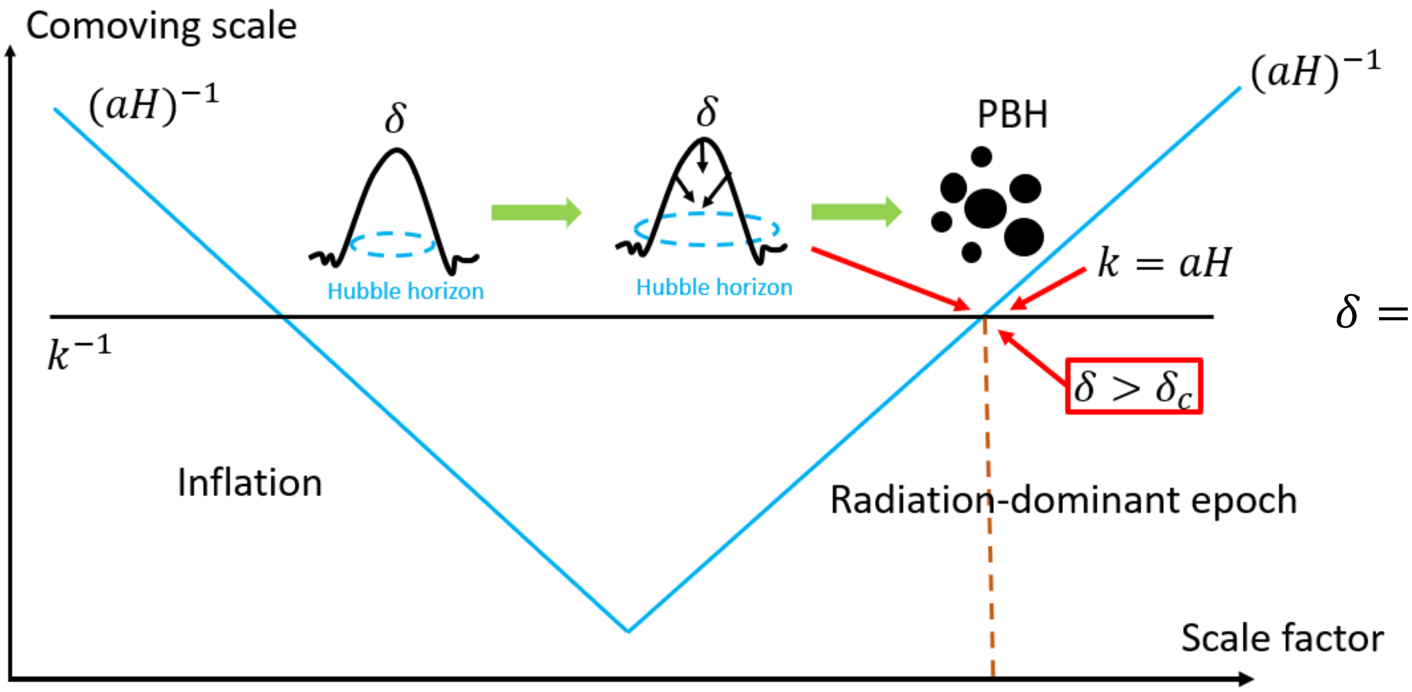
A statistical study on PBH binaries may help

PBH formation



$$\delta = \frac{\delta\rho}{\rho}$$

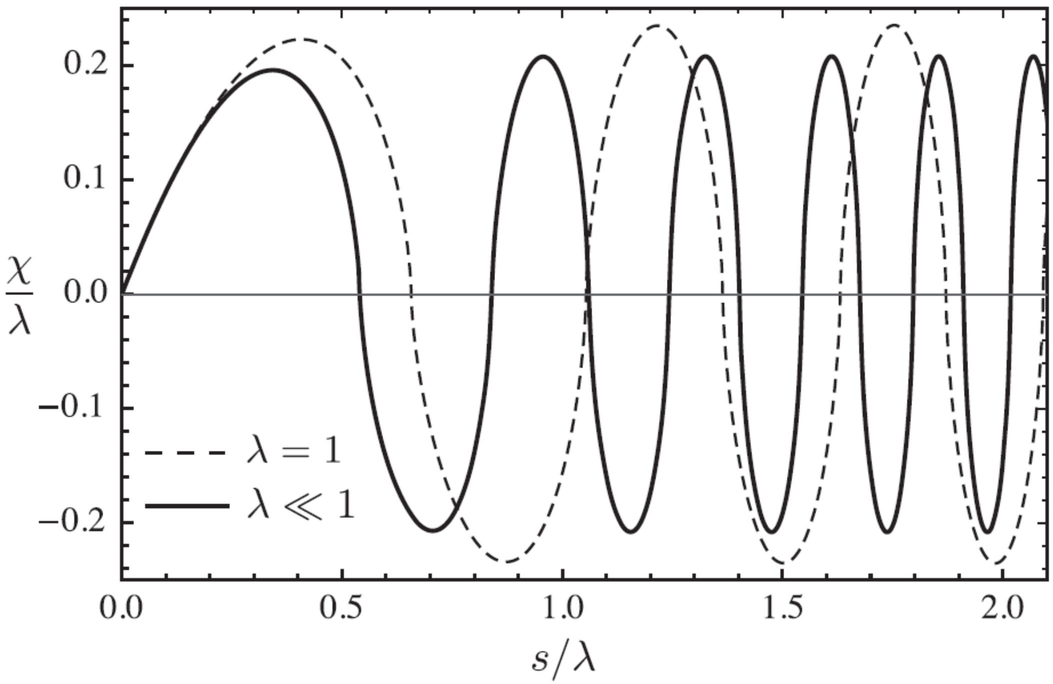
The primordial origin gives an identical primordial mass function

$$n(M) = \frac{f_{\rm PBH}}{\sqrt{2\pi}\sigma M} \exp\left[-\frac{\ln^2(M/M_{pk})}{2\sigma^2}\right]$$

PBH binary formation

The equation of proper separation r of two nearby PBHs with mass M is

$$\ddot{r} - \left(\dot{H} + H^2\right)r + \frac{2M}{r^2}\frac{r}{|r|} = 0$$



PBH binaries were formed with an identical probability distribution

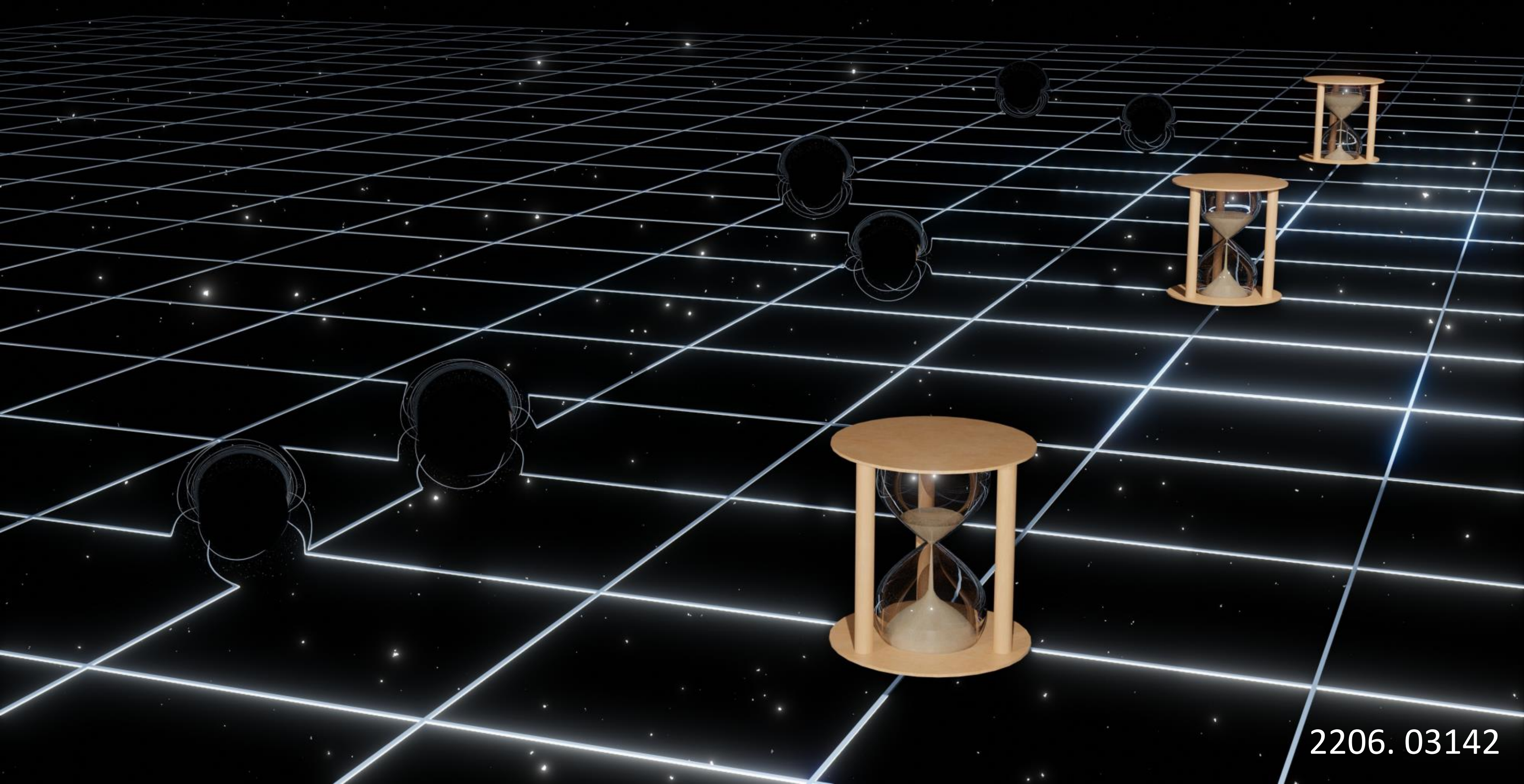
$$\frac{dP}{da\,de} = \frac{3}{4} f_{\rm PBH}^{3/2} \frac{a^{1/2}}{\bar{x}^{3/2}} \frac{e}{(1-e^2)^{3/2}}$$

Ali-Haïmoud, Yacine, Ely D. Kovetz, and Marc Kamionkowski. "Merger rate of primordial black-hole binaries." *Physical Review D* 96.12 (2017): 123523.

Primordial Black Hole Binaries as A Standard Timer

The initial probability distribution on a and e

$$\frac{dP}{dade} = \frac{3}{4} f_{\mathrm{PBH}}^{3/2} \frac{a^{1/2}}{\bar{x}^{3/2}} \frac{e}{(1-e^2)^{3/2}}$$



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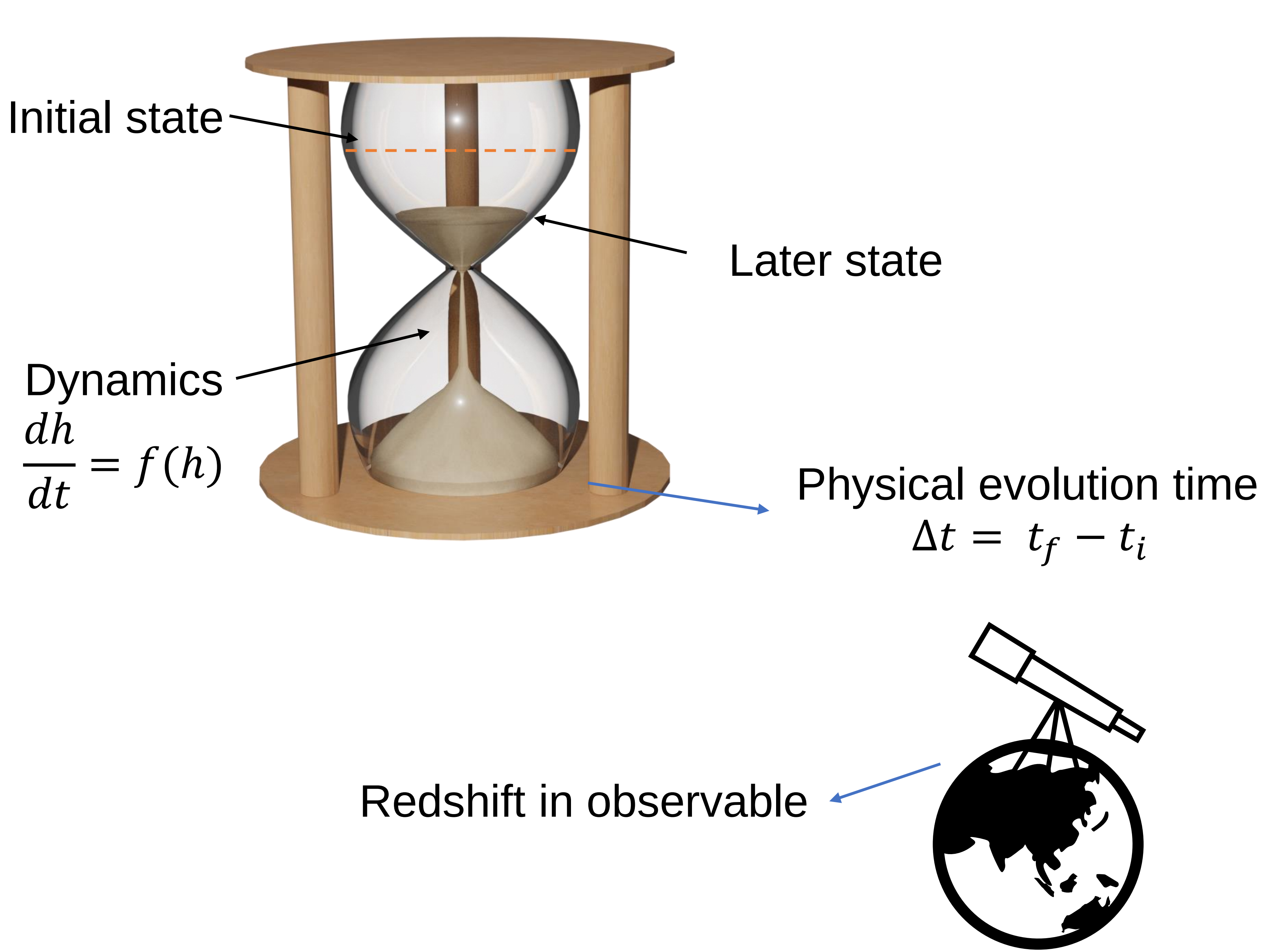


Initial state
Later state
Dynamics
$\frac{dh}{dt} = f(h)$
Physical evolution time
$\Delta t = t_f - t_i$
Redshift in observable

A single parameter standard timer

Initial state: Initial statistical distribution of dynamical systems

$$S(M;t_{\rm i}) = \frac{dN}{dM_{\rm i}}$$

Dynamics: time evolution of parameter in dynamical systems

$$\frac{dM}{dt} = -f(M) \Rightarrow \int_{M_{\rm i}}^{M_t} \frac{dM}{f(M)} = g(M_t) - g(M_{\rm i}) = -\Delta t$$

Later state: Statistical distribution of dynamical systems at physical time t

$$S(M;t) = \frac{dN}{dM_{\rm i}}\frac{dM_{\rm i}}{dM_t} = S(M;t_{\rm i})\frac{g'(M_t)}{g'(g^{-1}(g(M_t)+\Delta t))}$$

Observed state: Redshifted statistical distribution of dynamical systems detected at redshift z

$$S_o(M_z;t) = S_o(M_z;t_{\rm i})\frac{g'(M_z)}{g'(g^{-1}(g(M_z)+\Delta t_z))}$$

Redshift-time relation: Comparing the observed state with the initial state gives the redshift-time relation

$$S_o(M_z;t) \simeq \begin{cases} S(M;t_{\rm i})\dfrac{dM_{\rm i}}{dM_{\rm i}(z)} & , \ g(M_z) \gg \Delta t_z \\ S_o(g^{-1}(\Delta t_z);t_{\rm i})\dfrac{g'(M_z)}{g'(g^{-1}(\Delta t_z))}, & g(M_z) \ll \Delta t_z \end{cases}$$

A multi-parameter standard timer

Initial state: Initial statistical distribution of dynamical systems

$$S(\mathbf{M}; t_{\mathrm{i}}) = \frac{dN}{d^n \mathbf{M}_{\mathrm{i}}}$$

Dynamics: time evolution of parameter in dynamical systems

$$\frac{d\mathbf{M}}{dt} = -\mathbf{f}(\mathbf{M})$$

Later state: Statistical distribution of dynamical systems at physical time t

$$S(\mathbf{M}; t) = \frac{dN}{d^n \mathbf{M}_{\mathrm{i}}} \det \mathbf{J}(\mathbf{M}, \Delta t) = S(\mathbf{M}; t_{\mathrm{i}}) \det \mathbf{J}(\mathbf{M}, \Delta t)$$

$$\mathbf{J}_{ij} \equiv \partial M_i(t_{\mathrm{i}}) / \partial M_j(t)$$

Observed state: Redshifted statistical distribution of dynamical systems detected at redshift z

$$S_o(\mathbf{M}_z; t) = \frac{dN}{d^n \mathbf{M}_{\mathrm{i}}(z)} \det \mathbf{J}(\mathbf{M}_z, \Delta t_z)$$

How to extract the physical evolution time?

The evolution of probability distribution in PBH binaries

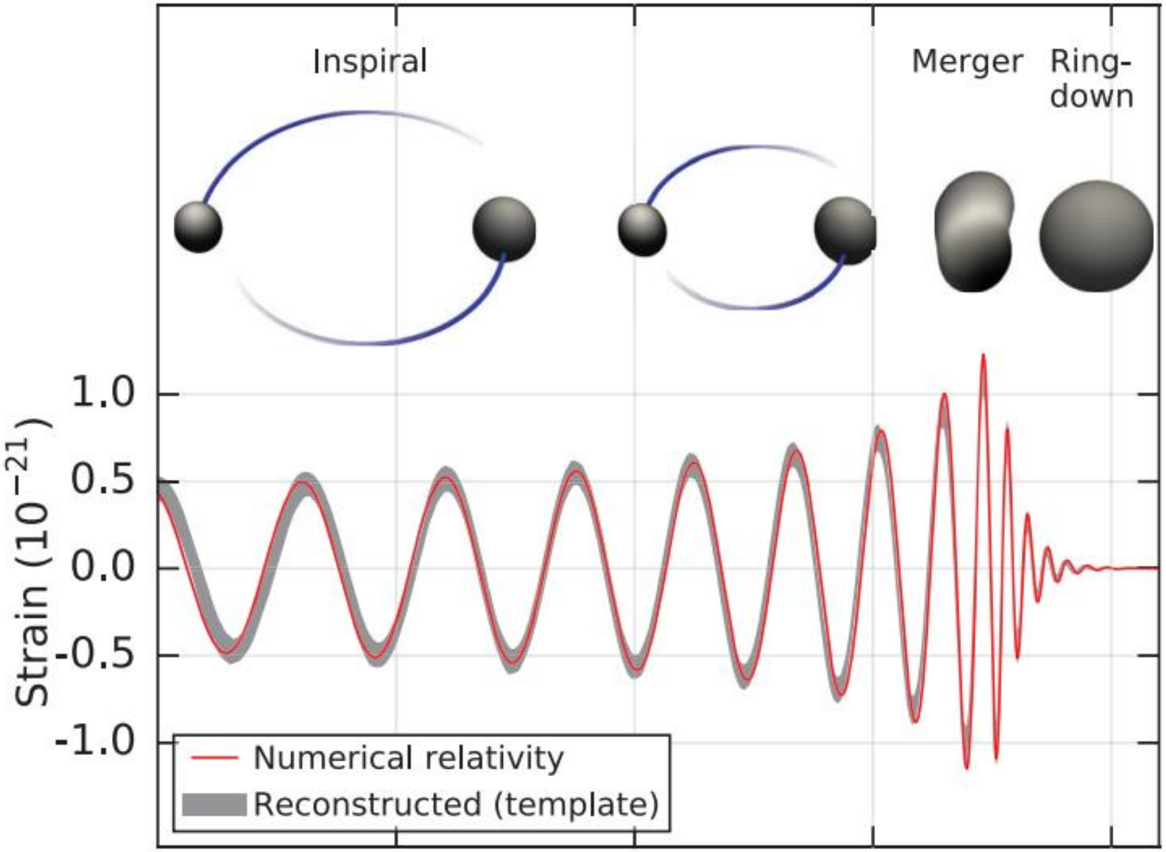
$$\frac{dP}{da_t de_t} = \frac{dP}{da_i de_i} \det \mathrm{J}(a, e, \Delta t)$$

$$\mathrm{J}(a, e, \Delta t) = \begin{pmatrix} \partial a_i/\partial a_t & \partial a_i/\partial e_t \\ \partial e_i/\partial a_t & \partial e_i/\partial e_t \end{pmatrix}$$

$$\frac{da}{dt} = -\frac{128}{5}\frac{G^3 M_{\mathrm{PBH}}^3}{c^5 a^3 (1-e^2)^{7/2}}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right)$$

$$\frac{de}{dt} = -\frac{608}{15}\frac{G^3 M_{\mathrm{PBH}}^3}{c^5 a^4}\frac{e}{(1-e^2)^{5/2}}\left(1 + \frac{121}{304}e^2\right)$$

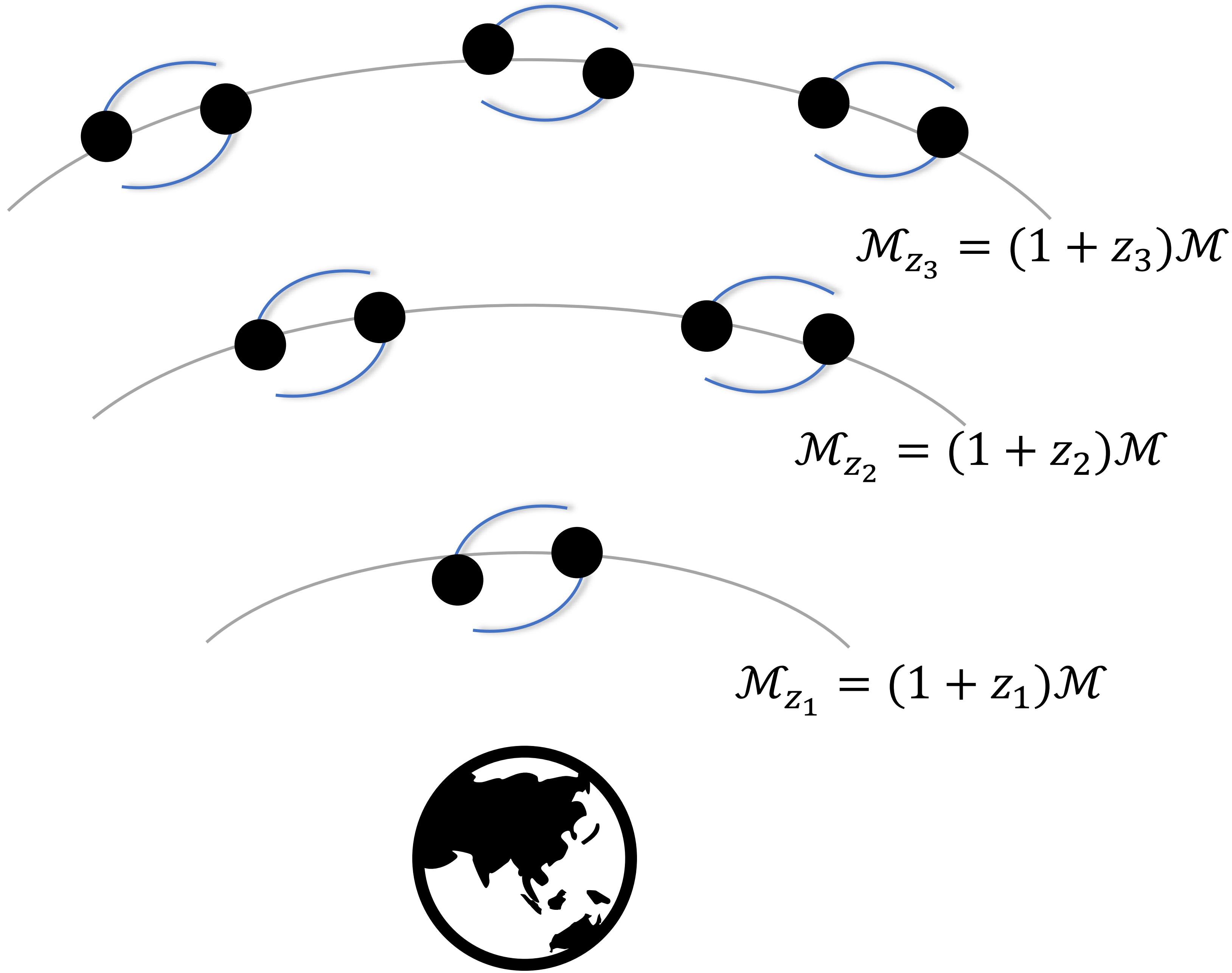
How to extract the redshift from the observable?

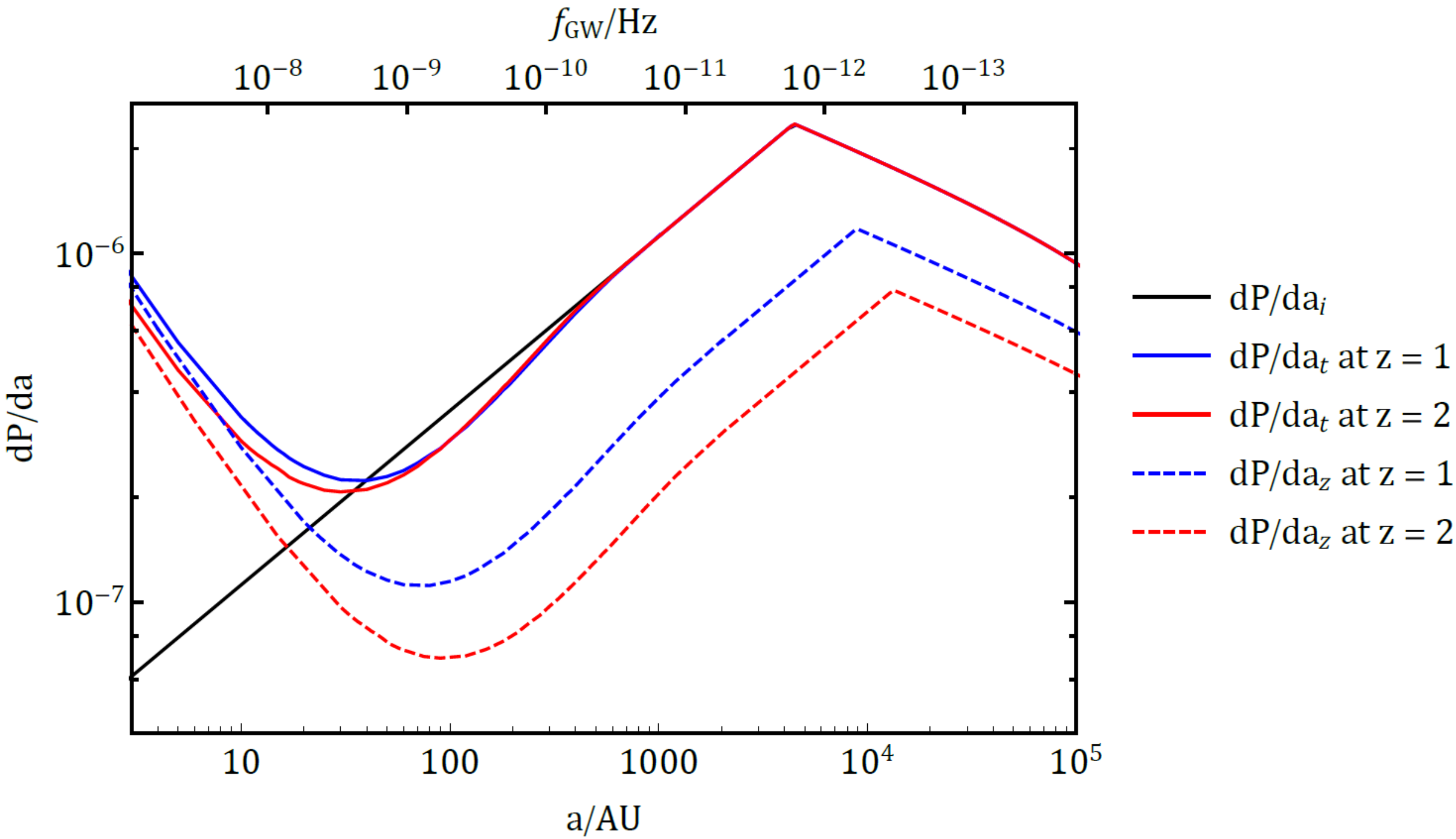


B. P. Abbott et al. Observation of Gravitational Waves from a Binary Black Hole Merger. Phys. Rev. Lett., 116(6):061102, 2016.

Redshifted Chirp Mass

$$\mathcal{M}_z = (1 + z)\mathcal{M}$$

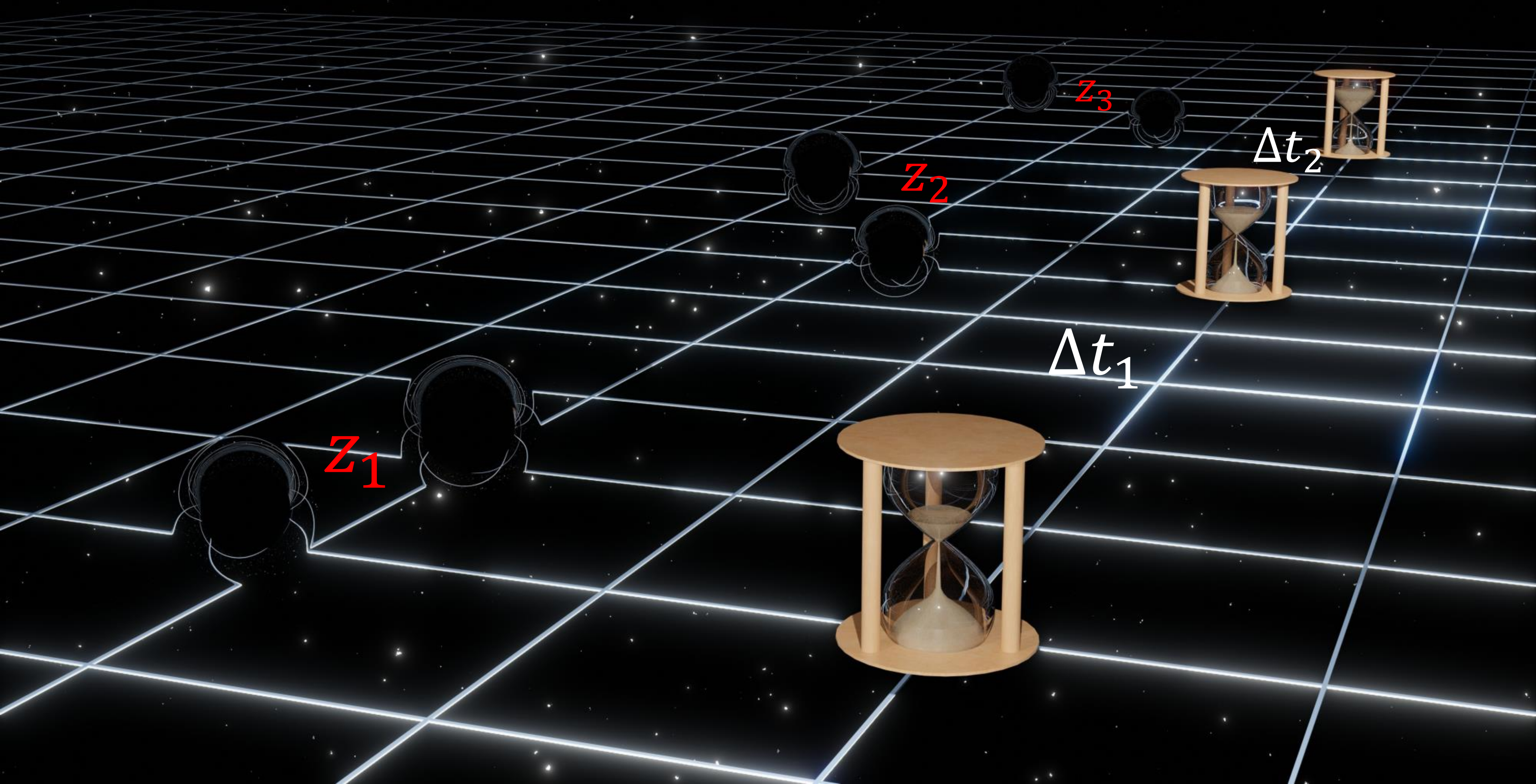

$\mathcal{M}_{z_3} = (1 + z_3)\mathcal{M}$
$\mathcal{M}_{z_2} = (1 + z_2)\mathcal{M}$
$\mathcal{M}_{z_1} = (1 + z_1)\mathcal{M}$



$$\frac{dP}{da_z} = \frac{1}{1+z}\frac{dP}{da_i}$$

$$\Delta t = \int_{z_1}^{z_2} \frac{dz}{H(z)(1+z)}$$

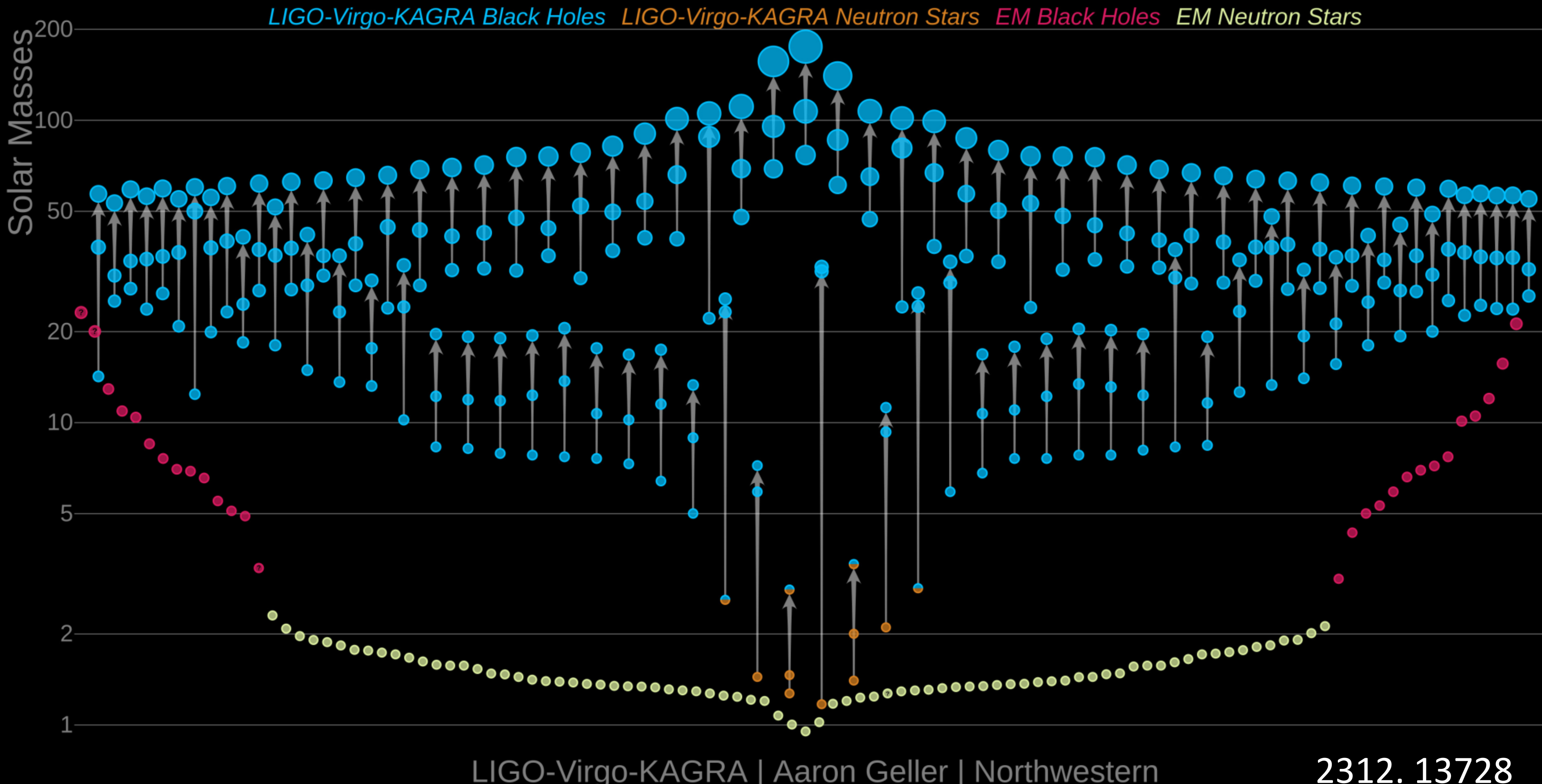
$$H(z) = H_0 \sqrt{\Omega_\gamma (1+z)^4 + \Omega_m (1+z)^3 + \Omega_\Lambda}$$



Merger rate of PBH binaries as a probe of Hubble parameter

PBH mass function

$$n(M) = \frac{1}{\sqrt{2\pi}\sigma M} \exp\left[-\frac{\ln^2(M/M_{pk})}{2\sigma^2}\right]$$



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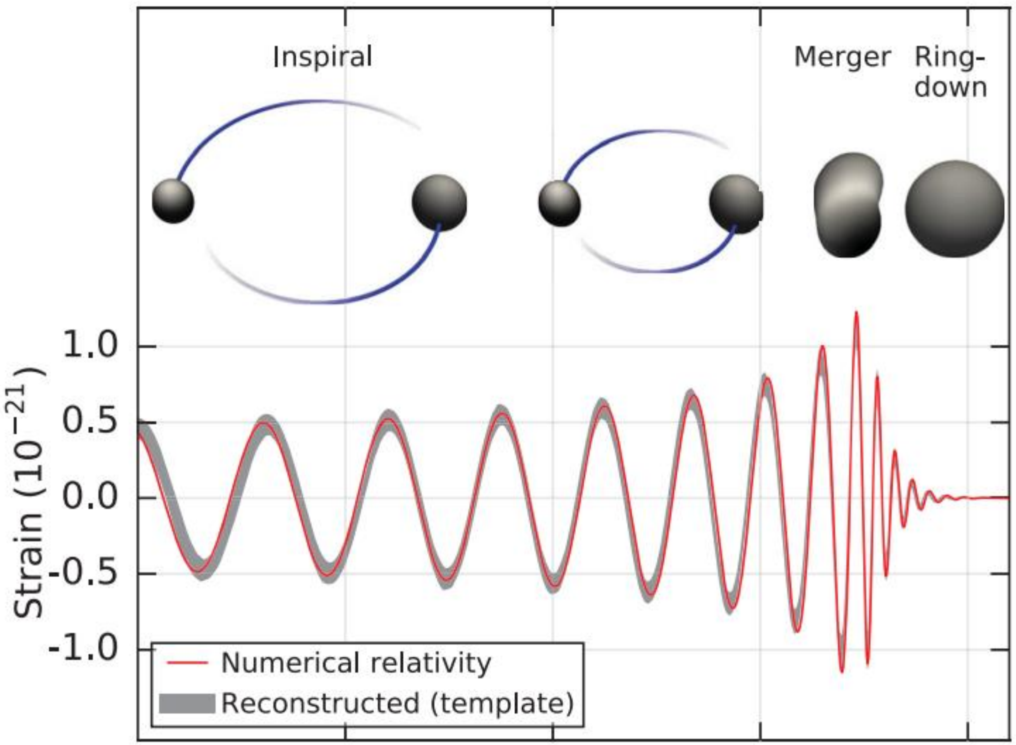
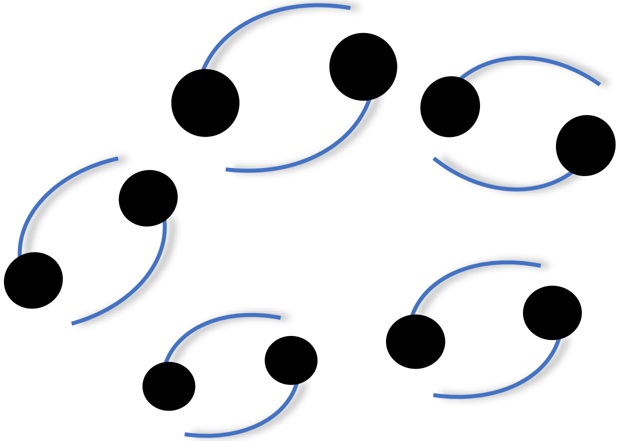


$$\mathcal{M}_z = (1+z)\mathcal{M}$$

$$d_L = \frac{1+z}{H_0}\int_0^z \frac{c}{E(z')}dz'$$

B. P. Abbott et al. Observation of Gravitational Waves from a Binary Black Hole Merger. Phys. Rev. Lett., 116(6):061102, 2016.

$$h(t) = \frac{4}{d_L(z)}\left(\frac{G\mathcal{M}_z}{c^2}\right)^{5/3}\left(\frac{\pi f(t)}{c}\right)^{2/3}\cos\Phi(t)$$



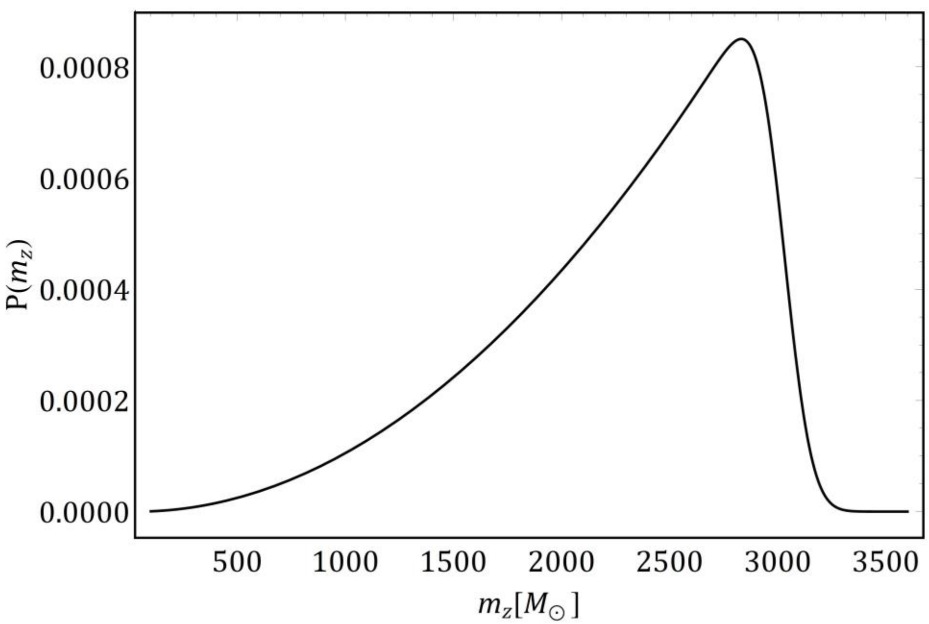
(m_z^i, d_L^i)

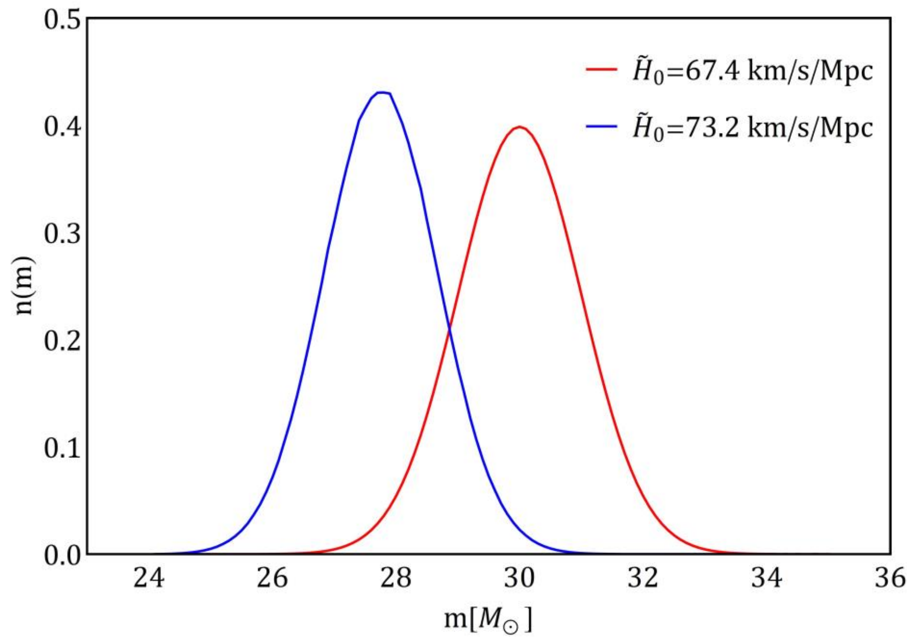
$$d_L^i = \frac{1+z_i}{H_0}\int_0^{z_i}\frac{c}{E(z')}dz'$$

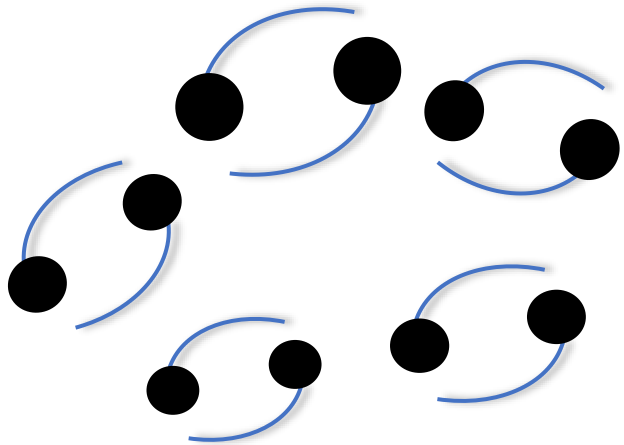
Assume a Hubble parameter $\widetilde{H}_0$

$$z_i = z(d_L^i; \widetilde{H}_0)$$

$$m^i = m_z^i/(1+z_i)$$







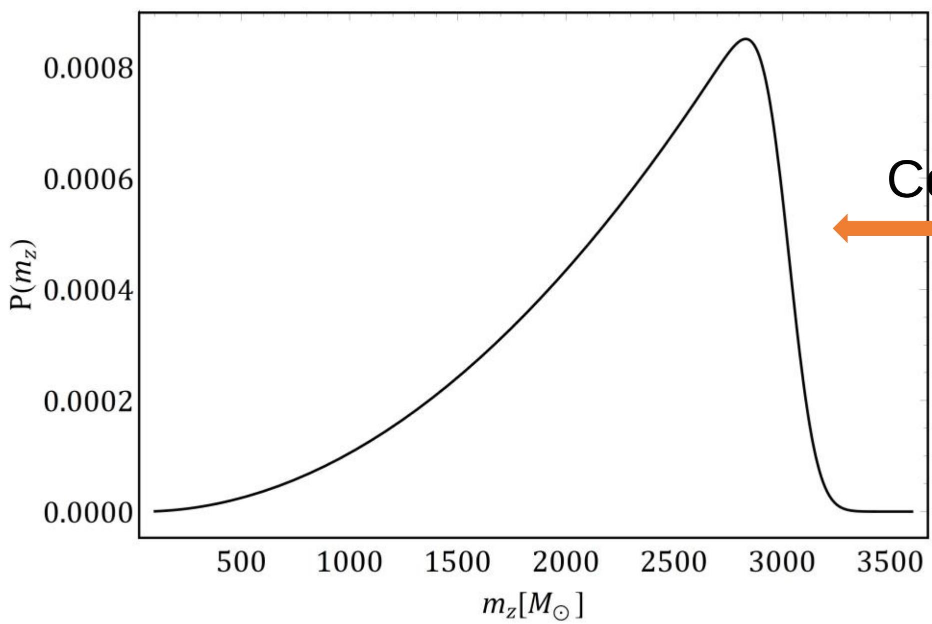
$$(m_z^i, d_L^i)$$

$$d_L^i = \frac{1+z_i}{H_0}\int_0^{z_i} \frac{c}{E(z')}\,dz'$$

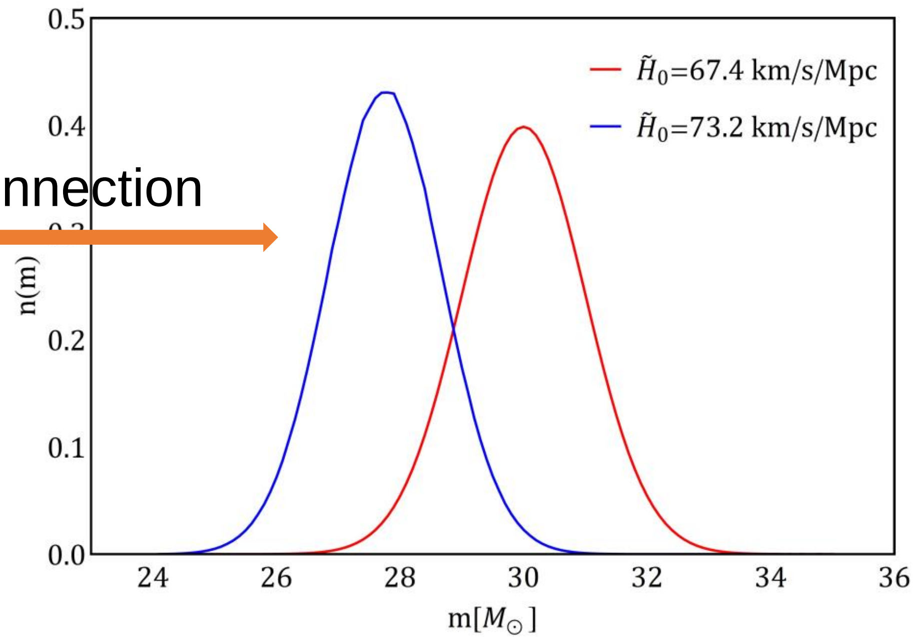
Assume a Hubble parameter $\widetilde{H}_0$

$$z_i = z(d_L^i; \widetilde{H}_0)$$

$$m^i = m_z^i/(1+z_i)$$



Connection



detectable window function

$$C(m_1^z, m_2^z) = \int_0^\infty \int_0^{\frac{m_1^z}{1+z}} \int_0^{\frac{m_2^z}{1+z}} n(m_1) n(m_2) W(m_1, m_2; z) p(z)\, dm_1 dm_2 dz$$

PBH mass function

redshift distribution

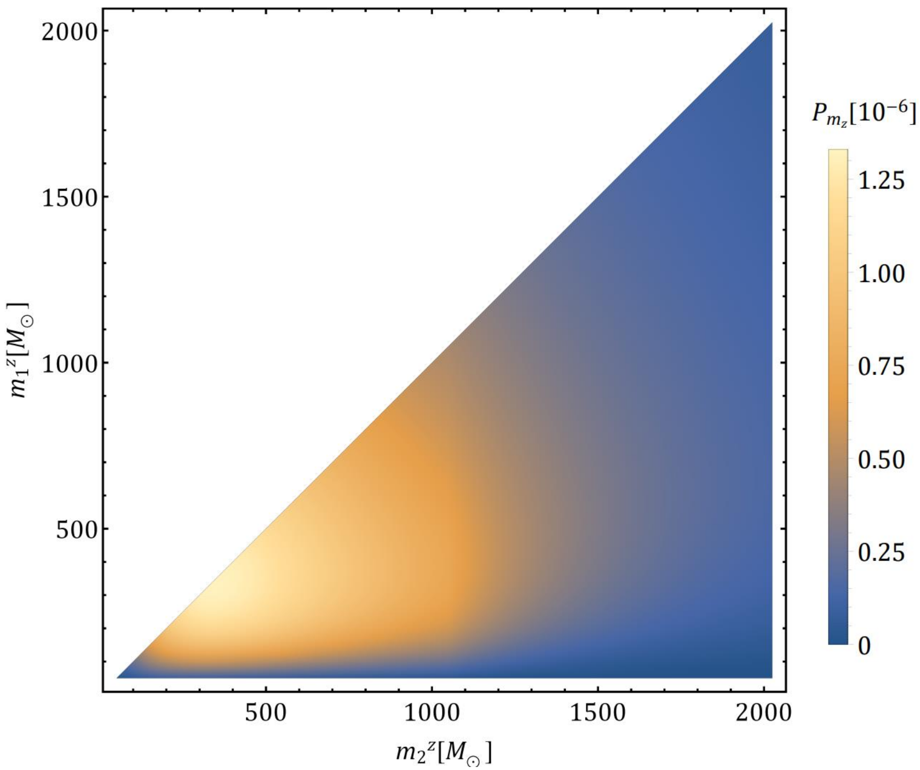
$$P(m_1^z, m_2^z) = \frac{\partial^2 C(m_1^z, m_2^z)}{\partial m_1^z \partial m_2^z}$$

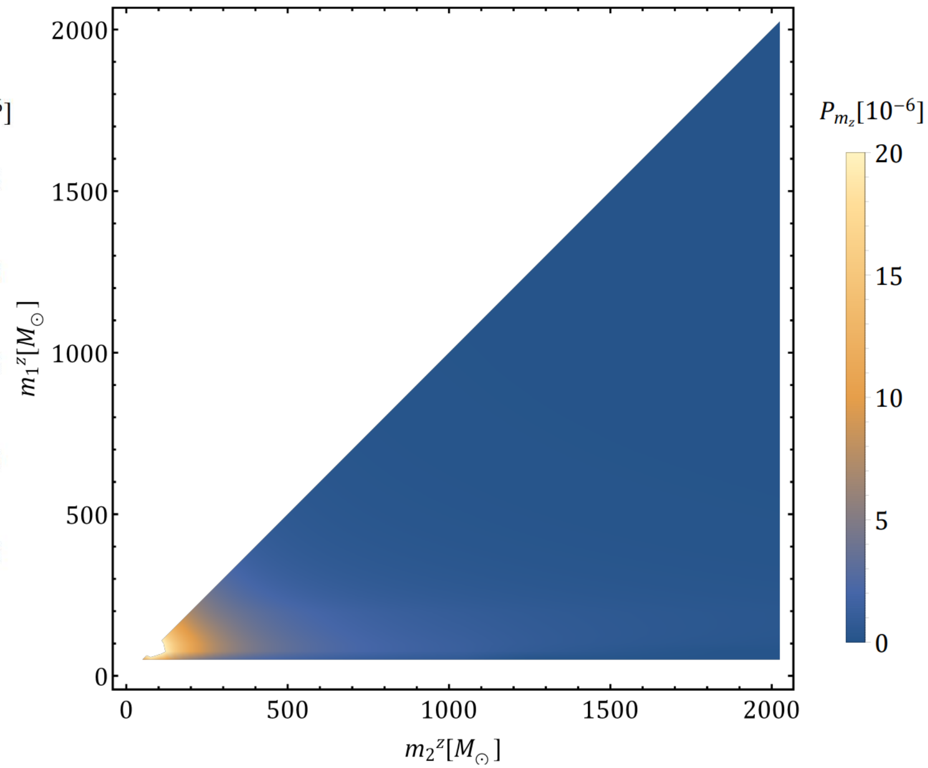
$$P(m_1^z, m_2^z) = \int_0^\infty n\left(\frac{m_1^z}{1+z}\right) n\left(\frac{m_2^z}{1+z}\right) W\left(\frac{m_1^z}{1+z}, \frac{m_2^z}{1+z}; z\right) \frac{p(z)}{(1+z)^2} dz$$

$$W(m_1, m_2; z) = \frac{N_{\rm obs}(m_1, m_2; z)}{N_{\rm tot}(m_1, m_2; z)} = \int_{a_{\rm min}}^{a_{\rm max}} \int_{e_{\rm min}}^{e_{\rm max}} P(a, e; z)\, dade$$

$$\mathrm{SNR} = \sqrt{4 \int_{f_{\rm min}}^{f_{\rm max}} \frac{\left|\tilde{h}(f)\right|^2}{S_n(f)} df} > 8 \quad \tilde{h}(f) = \sqrt{\frac{5}{24}} \frac{(G\mathcal{M}_z)^{5/6}}{\pi^{2/3} c^{3/2} d_L} f^{-7/6}$$

$$p(z) \propto \frac{\dot{n}(z)}{1+z} \frac{dV_c}{dz} \qquad \dot{n}(z) \propto \left(\frac{t(z)}{t_0}\right)^{-34/37}$$





$$n(m) = \frac{1}{\sqrt{2\pi}\sigma m}\exp\left[-\frac{\ln^2(m/m_{pk})}{2\sigma^2}\right]$$

$$n(m) = \frac{\alpha - 1}{M}\left(\frac{m}{M}\right)^{-\alpha}$$

$$P_O(m_1^z, m_2^z) = \int_0^\infty n_p\left(\frac{m_1^z}{1+z}\right) n_p\left(\frac{m_2^z}{1+z}\right) W\left(\frac{m_1^z}{1+z}, \frac{m_2^z}{1+z}; z\right) \frac{p(z)}{(1+z)^2}\, dz$$

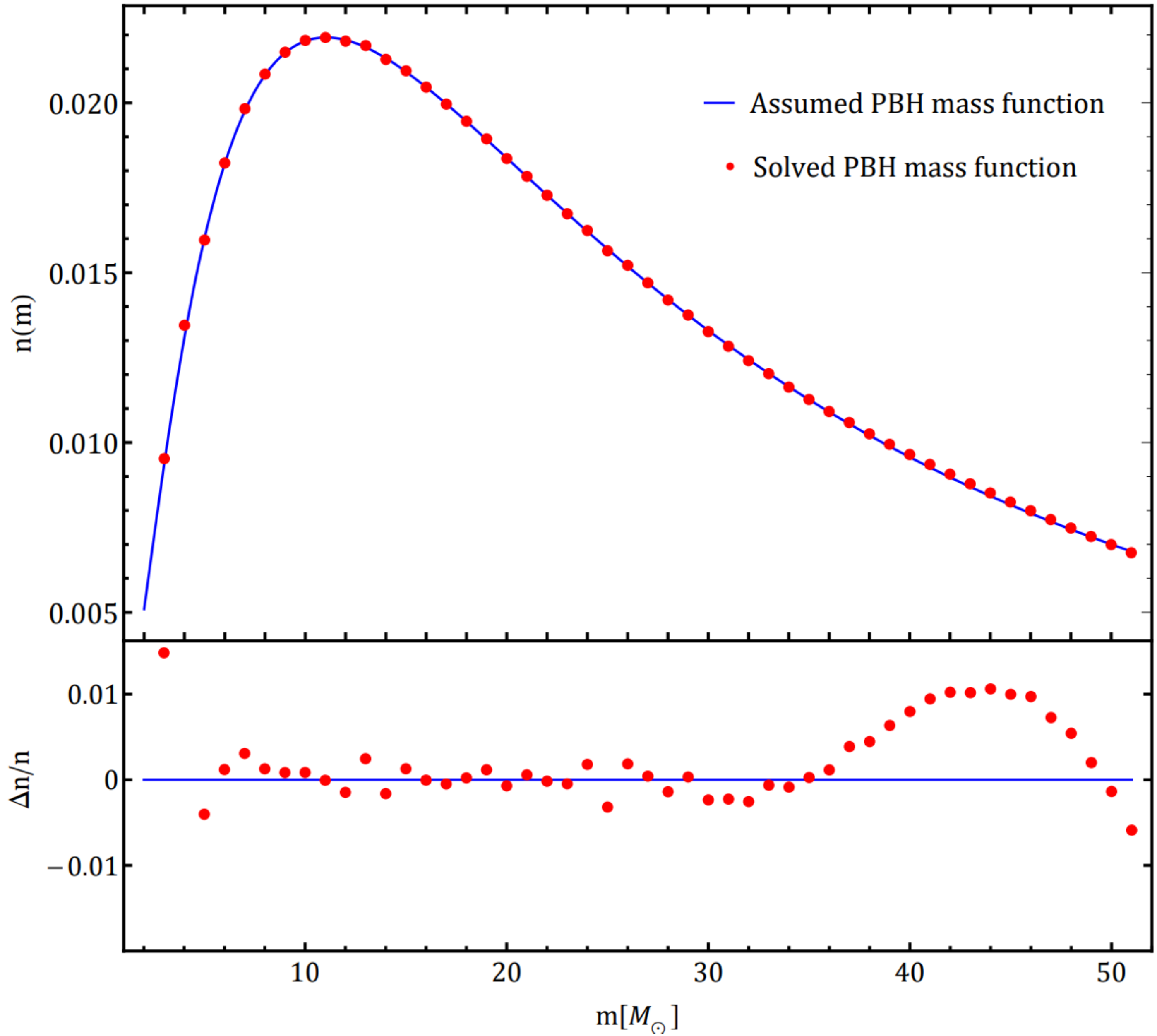
Gradient Descent Method

$$P_O(m_1^z, m_2^z) = \int_0^\infty n_p\left(\frac{m_1^z}{1+z}\right) n_p\left(\frac{m_2^z}{1+z}\right) W\left(\frac{m_1^z}{1+z}, \frac{m_2^z}{1+z}; z\right) \frac{p(z)}{(1+z)^2} dz$$

$$P_T(m_1^z, m_2^z) = \int_0^\infty n'\left(\frac{m_1^z}{1+z}\right) n'\left(\frac{m_2^z}{1+z}\right) W\left(\frac{m_1^z}{1+z}, \frac{m_2^z}{1+z}; z\right) \frac{p(z)}{(1+z)^2} dz$$

$$E(n) = \sqrt{\frac{\sum_{1\le i\le j, j=1}^{j=N} [P_T(m_1^z, m_2^z) - P_O(m_1^z, m_2^z)]^2}{(1+N)N/2}}$$

$$n_{k+1}(m_i) = n_k(m_i) - \gamma \frac{\partial E(n_k)}{\partial n_k(m_i)}$$



Assumed PBH mass function
Solved PBH mass function
n(m)
0.020
0.015
0.010
0.005
Δn/n
0.01
0
−0.01
10
20
30
40
50
m[$M_\odot$]

$$P_O(m_1^z, m_2^z) = \int_0^\infty n_z\left(\frac{m_1^z}{1+z}\right) n_z\left(\frac{m_2^z}{1+z}\right) W\left(\frac{m_1^z}{1+z}, \frac{m_2^z}{1+z}; z\right) \frac{p(z)}{(1+z)^2} dz$$

$$n_z(m_z) = n_{\mathrm{i}}(m_i)\frac{dm_{\mathrm{i}}}{dm_z} = n_{\mathrm{i}}(m_i) g(z, m_{\mathrm{z}})$$

$$\frac{dm}{dt} = 4\pi\lambda\rho_m \frac{G^2 m^2}{v_{\mathrm{eff}}^3}$$

$$P_O(m_1^z, m_2^z) = \int_0^\infty n_{\mathrm{i}}\left(\frac{m_1^z}{1+z}\right) n_{\mathrm{i}}\left(\frac{m_2^z}{1+z}\right) g(z, \frac{m_1^z}{1+z}) g(z, \frac{m_2^z}{1+z})$$
$$\times W\left(\frac{m_1^z}{1+z}, \frac{m_2^z}{1+z}; z\right) \frac{p(z)}{(1+z)^2} dz$$

How about $p(z)$?

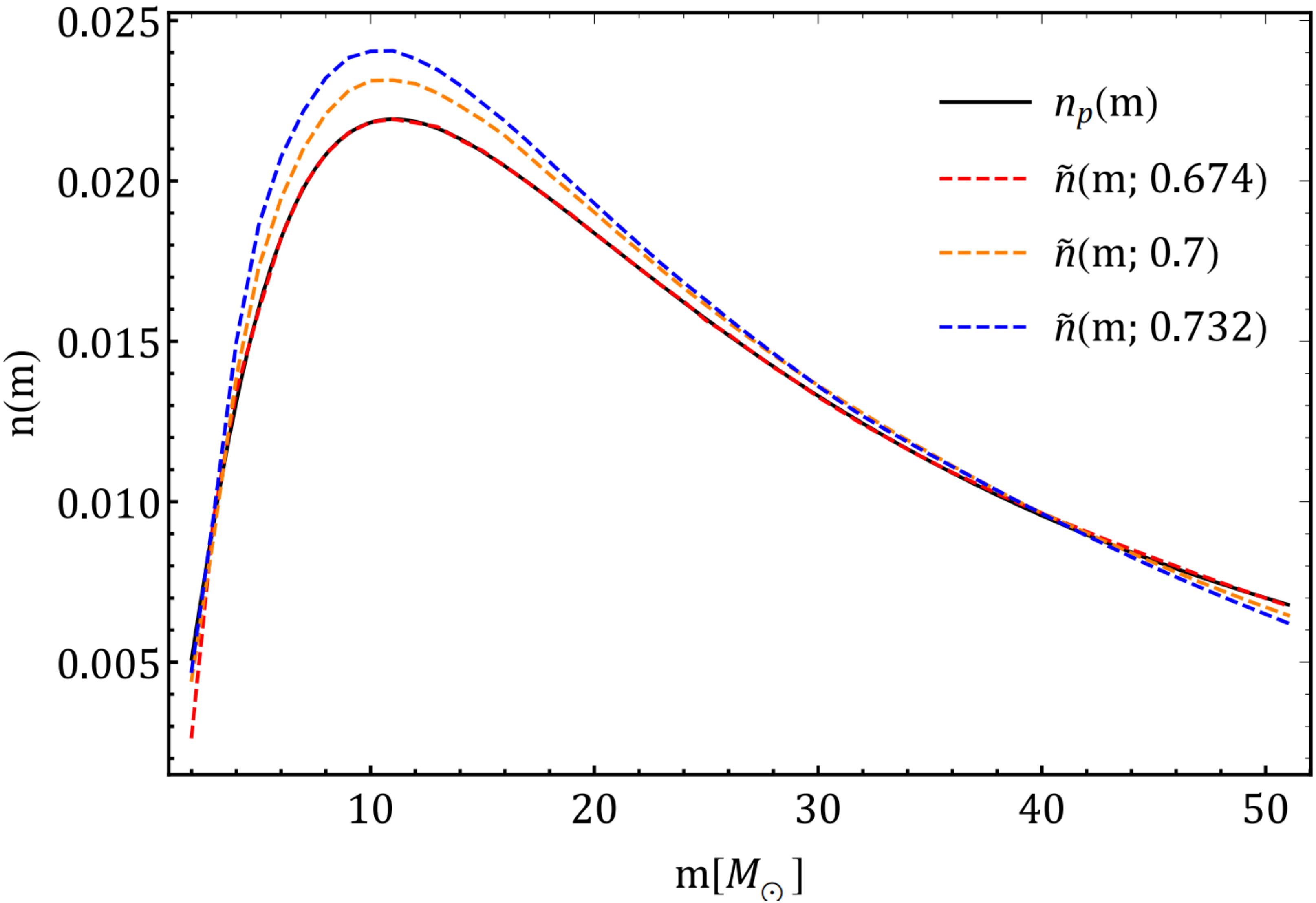
$$d_L^i = \frac{1 + z_i}{H_0} \int_0^{z_i} \frac{c}{E(z')} dz'$$

Assume a Hubble parameter $\widetilde{H}_0$

$$z_i = z(d_L^i; \widetilde{H}_0)$$

$$\Bigg\} \quad p(z; \widetilde{H}_0)$$

$$P_O(m_1^z, m_2^z) = \int_0^\infty \tilde{n}\left(\frac{m_1^z}{1+z}\right) \tilde{n}\left(\frac{m_2^z}{1+z}\right) W\left(\frac{m_1^z}{1+z}, \frac{m_2^z}{1+z}; z\right) \frac{p(z; \widetilde{H}_0)}{(1+z)^2} dz$$

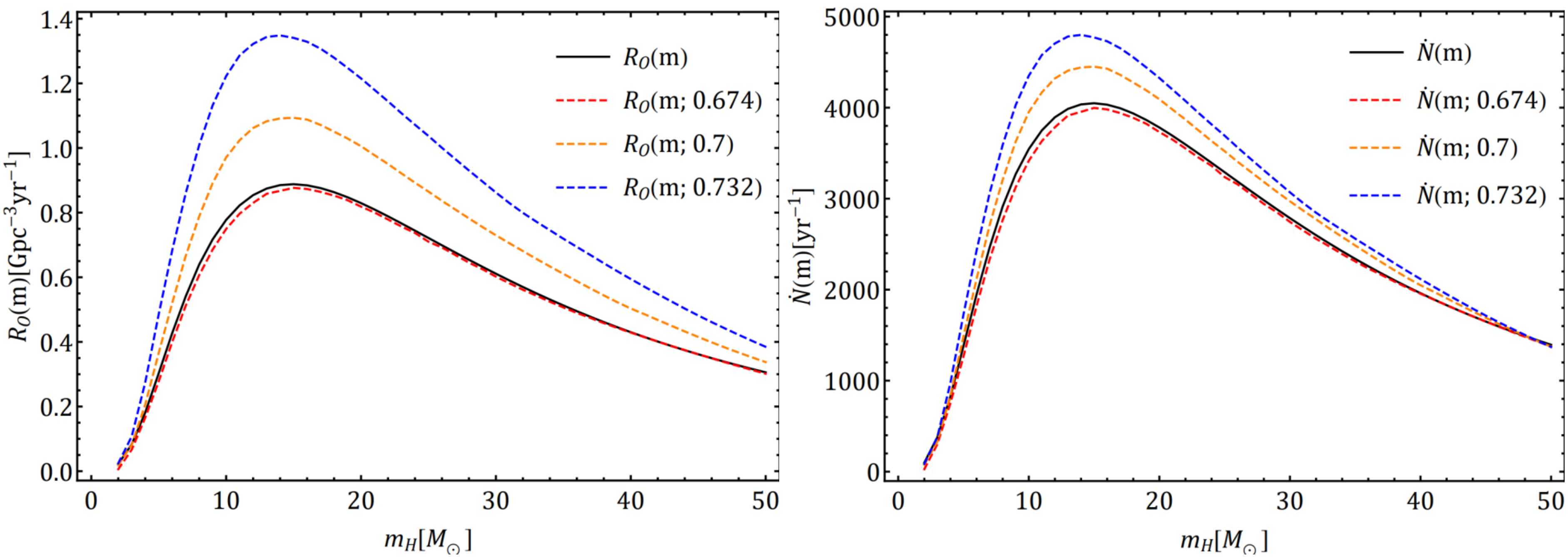


0.025
0.020
0.015
0.010
0.005
n(m)
10
20
30
40
50
m[$M_\odot$]
n_p(m)
$\tilde{n}$(m; 0.674)
$\tilde{n}$(m; 0.7)
$\tilde{n}$(m; 0.732)

However, we don’t know the PBH mass function currently.

However, we don’t know the PBH mass function currently.

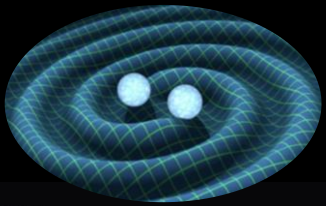
Another observable related with PBH mass function
Merger rate of PBH binaries

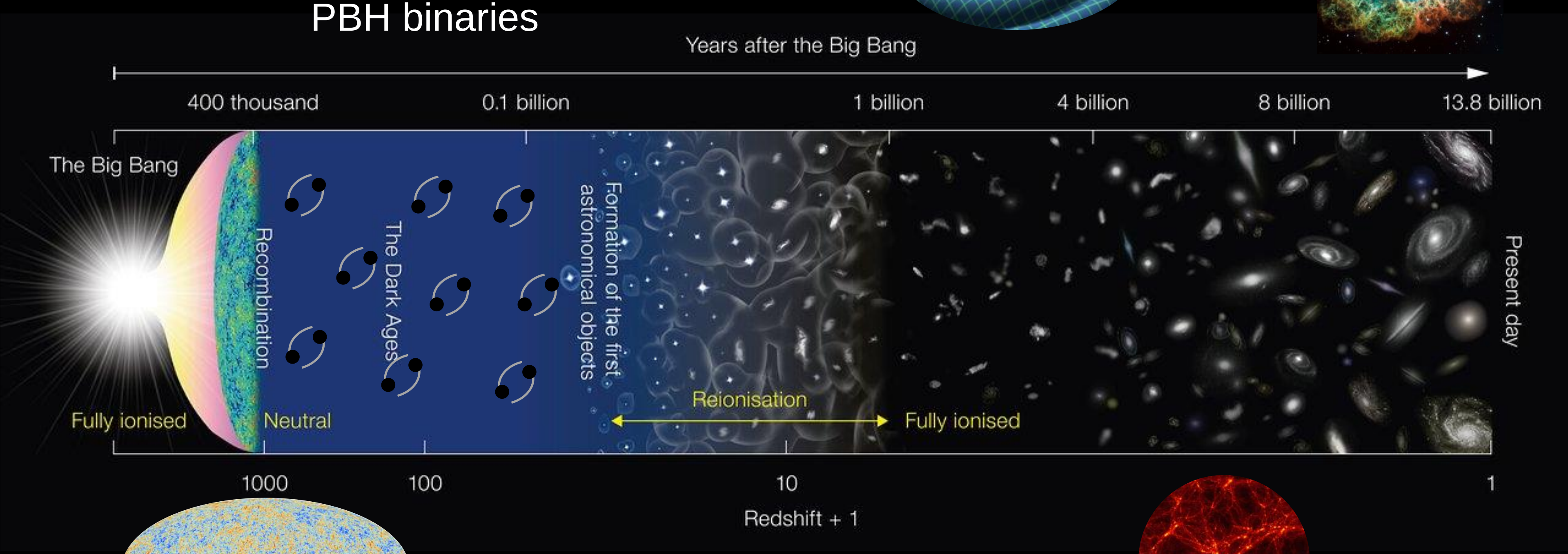


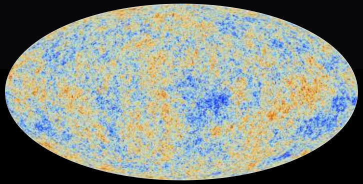
$$R_{ij} = \rho_{\text{PBH}} \min\left(\frac{n(m_i)}{m_i}, \frac{n(m_j)}{m_j}\right) \Delta_m \frac{dP}{dt}$$

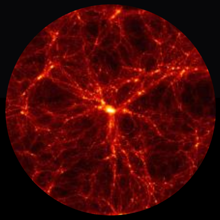
$$\dot{N} = \int_{z_{\min}}^{z_{\max}} \frac{R}{1+z} \frac{dV_c}{dz} dz$$

73.04 ± 1.04 km/s/Mpc







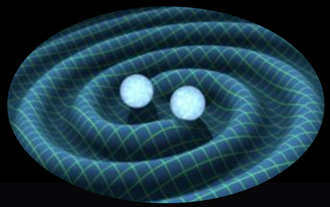


67.4 ± 0.5 km/s/Mpc

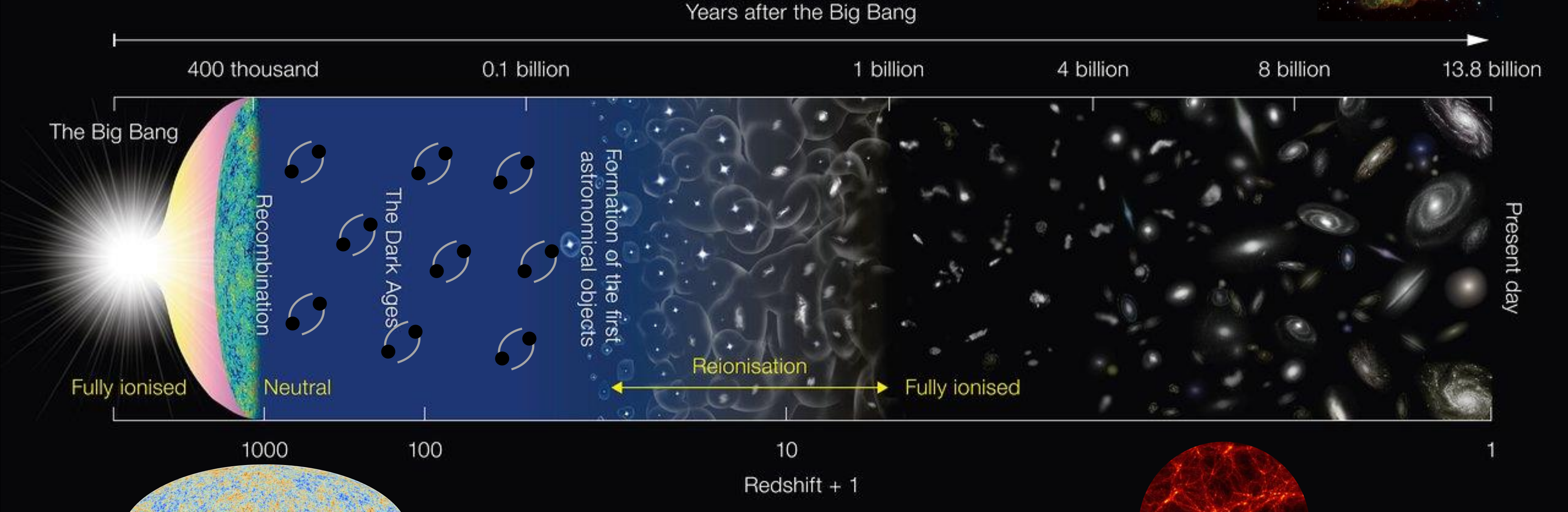
Image Credit: NAOJ

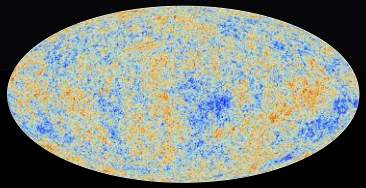
Thank you!

73.04 ± 1.04 km/s/Mpc



PBH binaries





67.4 ± 0.5 km/s/Mpc

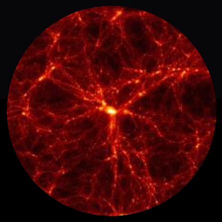
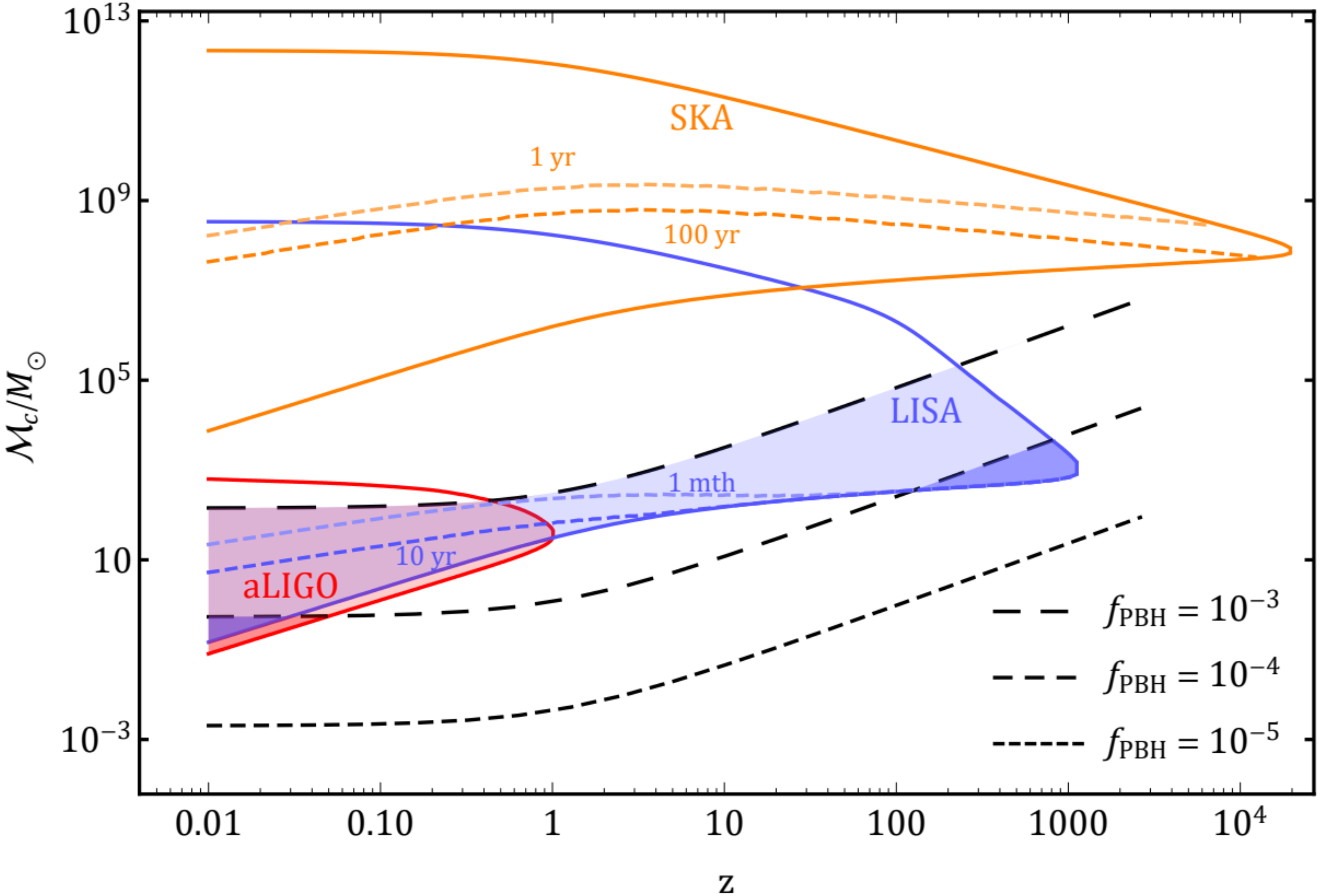


Image Credit: NAOJ


SKA
1 yr
100 yr
LISA
1 mth
10 yr
aLIGO
$f_{\rm PBH} = 10^{-3}$
$f_{\rm PBH} = 10^{-4}$
$f_{\rm PBH} = 10^{-5}$
$\mathcal{M}_c/M_\odot$
z
10^{13}
10^{9}
10^{5}
10
10^{-3}
0.01
0.10
1
10
100
1000
10^4

2011.13643